

Instantons in non-Cartesian coordinates

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Abstract

The explicit multi-instanton solutions by 'tHooft and Jackiw, Nohl & Rebbi are generalized to curvilinear coordinates. The idea is that a gauge transformation can notably simplify the expressions obtained after the change of variables. The gauge transform generates a compensating addition to the gauge potential of pseudoparticles. Singularities of the compensating field are irrelevant for physics but may affect gauge dependent quantities.

Introduction

The years that passed since the discovery of instantons [1] did not bring the answer to the question about the role of instantons in QCD [2, 3]. As far as confinement remains a puzzle all references to instantons at long scales are ambiguous. Indications may come from studies of instanton effects in phenomenological models. These could tell whether confinement may seriously affect pseudoparticles and *v. v.*

Common confinement models look most natural in non-Cartesian coordinate frames. The obvious choice for (spherical) bags are 3+1-cylindrical, *i. e.* 3-spherical+time, coordinates while strings would prefer 2+2-cylindrical (2+1-cylindrical+time) geometry. Nevertheless up till now instantons were usually discussed in the Cartesian frame (that was ideal in vacuum). In the present work we shall generalize to curvilinear coordinates the multi-instanton solutions by 'tHooft and Jackiw, Nohl & Rebbi [4] and simplify the

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formulae by the gauge transformation. I would expect that the procedure makes sense for the AHDM [5] solution and other topological configurations as well.

We shall start from reminding the basics of curvilinear coordinates in Sect. 1.1 where the Levi-Civita connection and the spin connection are described. In Sect. 1.2 we introduce the multi-instanton solutions known explicitly. In Sect. 2 we shall rewrite these formulae in non-Cartesian coordinates and propose the gauge transform that makes them compact. The price will be the appearance of the addition to the original field that we shall call the *compensating* gauge connection. The example of the $O(4)$ -spherical coordinates is sketched in Sect. 3. Singularities of the gauged solution are discussed in Sect. 4. The last part summarizes the results.

1 Basics

1.1 Curvilinear coordinates

We shall consider flat 4-dimensional euclidean space-time that may be parametrized either by a set of Cartesian coordinates x^μ or by curvilinear ones called q^α . The q -frame is characterized by the metric tensor $g_{\alpha\beta}(q)$ so that:

$$ds^2 = dx_\mu^2 = g_{\alpha\beta}(q) dq^\alpha dq^\beta. \quad (1)$$

Now the partial derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ must be replaced by the covariant ones D_α . For example the derivative of a contravariant vector A^β is:

$$D_\alpha A^\beta = \partial_\alpha A^\beta + \Gamma_{\alpha\gamma}^\beta A^\gamma. \quad (2)$$

The function $\Gamma_{\beta\gamma}^\alpha$ is called the **Levi-Civita connection**. It can be expressed in terms of the metric tensor and its inverse $g^{\alpha\beta}$:

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial q^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial q^\beta} - \frac{\partial g_{\beta\gamma}}{\partial q^\delta} \right). \quad (3)$$

Often it is helpful to use instead of $g_{\alpha\beta}$ the set of four vectors e_α^a called the vierbein:

$$g_{\alpha\beta}(q) = \delta_{ab} e_\alpha^a(q) e_\beta^b(q). \quad (4)$$

Multiplication by e_α^a converts coordinate (Greek) indices into the vierbein (Latin) ones,

$$A^a = e_\alpha^a A^\alpha. \quad (5)$$

Covariant derivatives of quantities with Latin indices are defined in terms of the **spin connection** $R_{\alpha b}^a(q)$,

$$D_\alpha A^a = \partial_\alpha A^a + R_{\alpha b}^a A^b. \quad (6)$$

The two connections $\Gamma_{\alpha\delta}^\beta$ and $R_{\alpha b}^a$ are related to each other as follows (note that both $g_{\alpha\beta}$ and e_α^a are covariantly constant by construction):

$$R_{\alpha b}^a = e_\beta^a \partial_\alpha e_b^\beta + e_\beta^a \Gamma_{\alpha\gamma}^\beta e_b^\gamma = e_\beta^a (D_\alpha e^\beta)_b. \quad (7)$$

Decomposition (4) of the metric into vierbeins is not unique. It is defined up to orthogonal transformations of the vierbein with respect either to the Greek or to Latin indices. Two vierbeins of the same orientation (that means those with the same signs of $\det ||e_\alpha^a||$) are related by a simple $O(4)$ rotation. Orientation of the vierbein can be changed by reversing one of its components, say, $e_1^a \rightarrow -e_1^a$.

It is convenient to use the freedom in order to make some components of vierbein zero. In general e_α^a is a 4×4 -matrix. However if the coordinate frame is orthogonal¹ and the metric tensor may be diagonalized,

$$g_{\alpha\beta}(q) = G(q) \delta_{\alpha\beta}, \quad (8)$$

then one can use the diagonal vierbein

$$e_\alpha^a = \sqrt{G(q)} \delta_\alpha^a. \quad (9)$$

We shall call it the *natural* vierbein. Only four of its components are not zero. From here on we shall assume that the curvilinear system is orthogonal so that the natural vierbein exists.

¹This class incorporates many coordinate systems including spherical, cylindric, parabolic, elliptical and others.

1.2 Instantons

We shall discuss the pure euclidean Yang-Mills theory with the $SU(2)$ gauge group. The vector potential is $\hat{A}_\mu = \frac{1}{2}\tau^a A_\mu^a$ where τ^a are the Pauli matrices. The (Cartesian) covariant derivative in fundamental representation is $D_\mu = \partial_\mu - i \hat{A}_\mu$, and the action has the form:

$$S = \int \frac{\text{tr } \hat{F}_{\mu\nu}^2}{2e^2} d^4x = \int \frac{\text{tr } \hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta}}{2e^2} \sqrt{g} d^4q. \quad (10)$$

where $g = \det ||g_{\alpha\beta}||$ and e is the coupling constant. The formula for the gauge field strength $\hat{F}_{\alpha\beta}$ is universal:

$$\hat{F}_{\alpha\beta}(\hat{A}) = \partial_\alpha \hat{A}_\beta - \partial_\beta \hat{A}_\alpha - i [\hat{A}_\alpha, \hat{A}_\beta]. \quad (11)$$

The action is invariant under gauge transformations

$$\hat{A}_\mu \rightarrow \hat{A}_\mu^\Omega = \Omega^\dagger \hat{A}_\mu(x) \Omega + i \Omega^\dagger \partial_\mu \Omega, \quad (12)$$

where Ω is a unitary 2×2 matrix, $\Omega^\dagger = \Omega^{-1}$.

The field equations have selfdual ($F_{\mu\nu} = \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} F^{\lambda\sigma}$) solutions known as instantons. It happens that all explicit multi-instanton solutions known by now are described by the same formula:

$$\hat{A}_\mu(x) = -\frac{\hat{\eta}_{\mu\nu}^-}{2} \partial_\nu \ln \rho(x), \quad (13)$$

where $\hat{\eta}_{\mu\nu}$ is the matrix version of the 'tHooft's η -symbol, [6]:

$$\hat{\eta}_{\mu\nu}^\pm = -\hat{\eta}_{\nu\mu}^\pm = \begin{cases} \tau^a \epsilon^{a\mu\nu}; & \mu, \nu = 1, 2, 3; \\ \pm \tau^a \delta^{\mu a}; & \nu = 4. \end{cases} \quad (14)$$

Depending on the choice of the scalar function $\rho(x)$ expression (13) describes either the higher selfdual configurations found by Jackiw, Nohl and Rebbi and 'tHooft's *Ansatz* [4] or instantons in singular and regular gauges (the latter requires the substitution $\hat{\eta}^- \rightarrow \hat{\eta}^+$ (23)).

Our aim is to generalize the solution (13) to curvilinear coordinates. We shall not refer to any particular form of $\rho(x)$ and the results will be applicable to all the cases.

2 Instantons in curvilinear coordinates

2.1 The question and the answer

PROBLEM It is not a big deal to transform the covariant vector \hat{A}_μ (13) to q -coordinates. However this makes the constant numerical tensor $\hat{\eta}_{\mu\nu}$ coordinate-dependent:

$$\hat{\eta}_{\mu\nu} \rightarrow \hat{\eta}_{\alpha\beta} = \hat{\eta}_{\mu\nu} \frac{\partial x^\mu}{\partial q^\alpha} \frac{\partial x^\nu}{\partial q^\beta} = \hat{\xi}_{ab} c_\alpha^a(q) c_\beta^b(q). \quad (15)$$

Here $\hat{\xi}_{ab}$ is a constant numerical matrix tensor that takes the place of $\hat{\eta}_{\mu\nu}$ in non-Cartesian coordinates:

$$\hat{\xi}_{ab} = \delta_a^\mu \delta_b^\nu \hat{\eta}_{\mu\nu}. \quad (16)$$

The former Cartesian vierbein² $c_\mu^a = \delta_\mu^a$ becomes a rather complicated 4×4 matrix $c_\alpha^a = \delta_\mu^a \frac{\partial x^\mu}{\partial q^\alpha}$ when expressed in q -coordinates. The question is if there is a way to replace it by the simpler natural vierbein e_α^a .

SOLUTION Let us chose the orientation of natural vierbein to coincide with that of c_α^a . Then these two can be rotated into each other. The $\hat{\eta}$ -symbols project spatial rotations onto the algebra of the $SU(2)$ gauge group. This means that rotations of the vierbein may be compensated by appropriate gauge transformations and

$$\Omega^\dagger \hat{\eta}_{\alpha\beta} \Omega = e_\alpha^a e_\beta^b \hat{\xi}_{ab}. \quad (17)$$

The gauge-rotated instanton field is the sum of the two pieces (12)

$$\hat{A}_\alpha^\Omega(q) = -\frac{1}{2} e_\alpha^a \hat{\xi}_{ab} e^{b\beta} \partial_\beta \ln \rho(q) + i \Omega^\dagger \partial_\alpha \Omega. \quad (18)$$

The first addend is almost traditional and does not depend on the Ω -matrix. The second one is entirely of geometrical origin and carries the information about the q -frame. We call it the **compensating connection** because it compensates the coordinate dependence of $\hat{\eta}_{ab} = e_a^\alpha e_b^\beta \hat{\eta}_{\alpha\beta}$ and reduces it to the constant $\hat{\xi}_{ab}$.

²The advantage of this vierbein is that it nullifies the spin connection.

So long we did not specify the duality of $\hat{\eta}$ -symbol. However the Ω -matrices and compensating connections $\hat{A}_\alpha^{\text{comp}\pm}$ are different for $\hat{\eta}^+$ and $\hat{\eta}^-$. In general they are respectively the selfdual and antiselfdual projections of the spin connection onto the gauge group:

$$\hat{A}_\alpha^{\text{comp}\pm} = i \Omega_\pm^\dagger \partial_\alpha \Omega_\pm = -\frac{1}{4} R_\alpha^{ab} \hat{\xi}_{ab}^\pm. \quad (19)$$

The last formula does not contain Ω that has dropped out of the final result. In order to write down the multi-instanton solution one needs only the vierbein and the associated spin connection.

2.2 Triviality of the compensating field.

The fact that the compensating connection $\hat{A}^{\text{comp}} = -\frac{1}{4} R_\alpha^{ab} \hat{\xi}_{ab}$ is a pure gauge (19) is specific to the flat space. It turns out that the field strength $\hat{F}_{\alpha\beta}(\hat{A}^{\text{comp}})$ is related to the Riemann curvature of the space-time $R_{\alpha\beta}{}^{\gamma\delta}$:

$$\hat{F}_{\alpha\beta}(\hat{A}^{\text{comp}\pm}) = -\frac{1}{4} R_{\alpha\beta}{}^{\gamma\delta} \hat{\xi}_{\gamma\delta}^\pm. \quad (20)$$

In the flat space $R_{\alpha\beta}{}^{\gamma\delta} = 0$ and consequently $\hat{F}_{\alpha\beta}(\hat{A}^{\text{comp}}) = 0$. Simple changes of variables $x^\mu \rightarrow q^\alpha$ do not generate curvature and \hat{A}^{comp} is a pure gauge. However in curved space-times the connection given by the last of expressions (19) may be nontrivial.

2.3 Duality and topological charge

As long as we limit ourselves to identical transformations the vector potential (18) must satisfy the classical field equations. However the duality equation looks differently in non-Cartesian frame. If written with coordinate indices it is:

$$\hat{F}_{\alpha\beta} = \frac{\sqrt{g}}{2} \epsilon_{\alpha\beta\gamma\delta} \hat{F}^{\gamma\delta}. \quad (21a)$$

Still it retains the familiar form in the vierbein notation:

$$\hat{F}_{ab} = \frac{1}{2} \epsilon_{abcd} \hat{F}^{cd}. \quad (21b)$$

The topological charge is given by the integral

$$q = \frac{1}{32\pi^2} \int \epsilon_{\alpha\beta\gamma\delta} \text{tr} \hat{F}^{\alpha\beta} \hat{F}^{\gamma\delta} d^4q, \quad (22a)$$

which in the vierbein notation becomes

$$q = \frac{1}{32\pi^2} \int \epsilon_{abcd} \text{tr} \hat{F}^{ab} \hat{F}^{cd} \sqrt{g} d^4q. \quad (22b)$$

The general expression for $\hat{F}_{\alpha\beta}$ in non-Cartesian frame is rather clumsy [7] but it simplifies for one instanton. The Cartesian vector potential in regular gauge is, ($r^2 = x_\mu^2$):

$$\hat{A}_\mu^I = -\frac{\hat{\eta}_{\mu\nu}^+}{2} \partial_\nu \ln \frac{\rho^2}{r^2 + \rho^2}. \quad (23)$$

The conjugated coordinate and Ω_+ gauge transformations convert it into

$$\hat{A}_\alpha^I = -\frac{1}{2} e_\alpha^a \hat{\xi}_{ab} e^{b\beta} \partial_\beta \ln \frac{\rho^2}{r^2 + \rho^2} + \hat{A}^{\text{comp}+}, \quad (24)$$

and the field strength becomes plainly selfdual:

$$\hat{F}_{ab}(\hat{A}^I) = -\frac{2\hat{\xi}_{ab}^+}{(r^2 + \rho^2)^2}, \quad (25)$$

This generalizes the regular gauge to any curvilinear coordinate system.

3 Example

We shall consider the instanton placed at the origin of the $O(4)$ -spherical coordinates. Those are the three polar angles and the radius: $q^\alpha = (\chi, \phi, \theta, r)$. The polar axis is aligned with x^1 and

$$x^1 = r \cos \chi; \quad (26a)$$

$$x^2 = r \sin \chi \sin \theta \cos \phi; \quad (26b)$$

$$x^3 = r \sin \chi \sin \theta \sin \phi; \quad (26c)$$

$$x^4 = r \sin \chi \cos \theta. \quad (26d)$$

The properly oriented natural vierbein for spherical coordinates is:

$$e_\alpha^a = \text{diag} (r, r \sin \chi \sin \theta, r \sin \chi, 1), \quad (27)$$

Now one may start from the vector potential (23) and step by step carry out the entire procedure. But to the straightforward calculation of the instanton part this requires computing $\Gamma_{\beta\gamma}^\alpha$, then R_α^{ab} and finally $\hat{A}^{\text{comp}+}$. The result is:

$$\hat{A}_\chi^I = \frac{\tau_x}{2} \left(\frac{r^2 - \rho^2}{r^2 + \rho^2} \right); \quad (28a)$$

$$\begin{aligned} \hat{A}_\phi^I &= -\frac{\tau_x}{2} \cos \theta + \frac{\tau_y}{2} \sin \chi \sin \theta \left(\frac{r^2 - \rho^2}{r^2 + \rho^2} \right) \\ &\quad + \frac{\tau_z}{2} \cos \chi \sin \theta; \end{aligned} \quad (28b)$$

$$\hat{A}_\theta^I = -\frac{\tau_y}{2} \cos \chi + \frac{\tau_z}{2} \sin \chi \left(\frac{r^2 - \rho^2}{r^2 + \rho^2} \right); \quad (28c)$$

$$\hat{A}_r^I = 0. \quad (28d)$$

The corresponding field strength is given by (25).

4 Singularities

Note that the vector field (28) is singular since neither \hat{A}_θ^I nor \hat{A}_ϕ^I go to zero at $\chi = 0, \pi$ and $\theta = 0, \pi$. As a result they change stepwise across the Cartesian x_1x_4 -plane. This singularity is produced by the compensating gauge transformation and must not affect observables. However it may tell on gauge variant quantities. We shall demonstrate that for the Chern-Simons number.

The topological charge (22) can be represented by the surface integral, $q = \oint K^\alpha dS_\alpha$ [2, 3]. Here

$$K^\alpha = \frac{\epsilon^{\alpha\beta\gamma\delta}}{16\pi^2} \text{tr} \left(\hat{A}_\beta \hat{F}_{\gamma\delta} + \frac{2i}{3} \hat{A}_\beta \hat{A}_\gamma \hat{A}_\delta \right). \quad (29)$$

Even though q is invariant K^α depends on gauge. Consider the Cartesian instanton in the $\hat{A}_4 = 0$ gauge. The two contributions to the topological

charge come from the $x_4 = \pm\infty$ hyperplanes, $q = N_{\text{CS}}(\infty) - N_{\text{CS}}(-\infty)$, and the quantity

$$N_{\text{CS}}(t) = \int_{x_4=t} K^4 dS_4 \quad (30)$$

is called the Chern-Simon number. Instanton is a transition between two 3-dimensional vacua with $\Delta N_{\text{CS}} = 1$.

Analysis of (28) reveals a striking resemblance with this case. By coincidence here again $\hat{A}_4 = 0$, (28d). This gives the idea to treat r like a time coordinate attributing the Chern-Simons number $N_{\text{CS}}(r)$ to the sphere of radius r . A naive expectation would be that $\Delta N_{\text{CS}} = N_{\text{CS}}(r)|_0^\infty$ gives the topological charge. However this is not true and $N_{\text{CS}}(r)|_0^\infty = \frac{1}{2}$. The second half of q is contributed by the singularities at $\theta = 0, \pi$. We see that the gauge transformation has affected the distribution of N_{CS} .

We conclude that in our approach gauge variant quantities depend on the choice of coordinate system and may be localized at the singularities of the Ω -transform. This may be another promising possibility to simplify calculations with the help of curvilinear coordinates.

Summary

We have shown how the explicit (*multi*-)instanton solutions can be generalized to curvilinear coordinates. The gauge transformation converts the vierbein-dependent $\hat{\eta}_{ab}$ -symbol into the constant numerical matrix $\hat{\xi}_{ab}$. The gauge potential is a sum of the instanton part and the compensating gauge connection (18).

There is no need to know the manifest form of the gauge transform in order to calculate the compensating connection. The computation proceeds as follows:

1. One starts from calculating the Levi-Civita connection $\Gamma_{\beta\gamma}^\alpha$, (3).
2. Covariant differentiation of the vierbein leads to the spin connection R_α^{ab} , (7).
3. Convolution of the spin connection with the appropriate $\hat{\xi}_{ab}$ gives the compensating gauge potential, (19).

The advantage of our solution is that it is constructed directly of geometrical quantities, *i. e.* of the vierbein and the spin connection. I hope that it may be useful for studies of instanton effects in geometrically nontrivial phenomenological models. More details may be found in [7].

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